ASYMPTOTIC OPTIMIZATION OF CONSUMPTION IN A MODEL OF TWO-SECTOR ECONOMY

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In what proportions must be distributed the labor and the capital resources between the two sectors of economy — the production of the means of production and the production of the objects of consumption — in order to maximize the production of objects of consumption per capita of the population? In this article we describe a two-sector model of economy, within whose framework we give an answer to the above question (it is clear that the result has a purely model character and is not a recommendation for actual economy). The result is as follows: In the first place, from the exponential trajectories of development of the economy we select the "golden" trajectory on which the per capita consumption is greater than on other exponential trajectories. (This trajectory can be described with the help of "the golden rule" in economic terms without using the concrete form of the production functions.) Secondly we show that the average consumption on an arbitrary (not necessarily exponential) trajectory cannot exceed appreciably the consumption on the golden trajectory on an arbitrarily large interval of time. Thirdly, we show that from an arbitrary initial state we can asymptotically approximate to the golden trajectory.

1. THE GOLDEN RULE.

GEOMETRY OF EXPONENTIAL TRAJECTORIES

We start with the formulation of the "golden rule" ([1]). For the optimal development of economy a) *the investments in the first sector are equal to the profit connected with the capital of the first sector, and the investments in the second sector are equal to the profit connected with the labor of the first sector;* b) *the labor resources are divided between the first and the second sectors in proportion to the profit connected with the capital of the second sector and the profit connected with the labor of the second sector.*

In this section, we describe a two-sector model of economy with continuous time. We give geometrical description of the exponential trajectories of the model. It is shown that the maximum consumption is attained on the trajectory satisfying the golden rule, among the exponential trajectories.

1.1. A Two-Sector Model of Economy

Let us consider the two-sector model of economy

$$\dot{K}_1 = I_1 - \mu K_1, \tag{1}$$

$$\dot{K}_2 = I_2 - \mu K_2, \tag{2}$$

$$I_1 + I_2 \leqslant F_1(K_1, L_1), \tag{3}$$

$$L_1 + L_2 = e^{pt}, \tag{4}$$

$$C = F_2(K_2, L_2). \tag{5}$$

Here K_1, L_1, K_2, and L_2 are the capital and the labor resources in the first and the second sectors, I_1 and I_2 are the investments in the first and the second sectors, F_1 and F_2 are the production functions defining the output in each sector (these functions are assumed to be smooth and convex); μ is the amortization coefficient, p is the rate of growth of the total labor resources, and C is the total consumption. Let c denote the per capita consumption (or, in short, simply consumption): c = C/L (namely, we will optimize this quantity). The inequality (3) means that the sum of investments does not exceed the production of the first sector (the trajectories, on which this inequality becomes an identical equality, will be said to be taut). The first two equations mean that the investments in each sector go to

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increase capital, whereas the μ-th part of the capitals grows old and is taken out from production. The set of the nonnegative functions K_j, L_j, I_j, $j = 1, 2$, that satisfy the system (1)-(5) is a trajectory of the model.

For each sector of economy $j = 1, 2$, it is convenient to introduce relative economic characteristics by dividing the characteristics of each sector by the number of persons engaged in it. Let k_j denote the fund allocation in each sector (is equal to K_j/L_j), $f_j(k_j)$ denote the productivity of labor [is equal to $F_j(K_j, L_j)/L_j$], and i_j denote the reduced investment (is equal to I_j/L_j).

As is often accepted, we will assume that the productivity of labor f_j in each sector is a strictly convex (i.e., $f_j'' < 0$) monotonically increasing function.

Proposition 1. In the considered model there exists a two-parametric family of taut economic trajectories (i.e., trajectories in which all the functions are proportional to an exponent with power p and $I_1 + I_2 = F_1$). Such a trajectory is determined by each preassigned pair of nonnegative fund allocations (k_1^0, k_2^0), where k_1^0 does not exceed the positive root of the equation $f_1(k) = (\mu + p)k$. The following relations are fulfilled on an exponential trajectory:

$$\frac{I_2}{I_1} = \frac{f_1(k_1^0) - (\mu + p)k_1^0}{(\mu + p)k_1^0},$$

$$\frac{L_2}{L_1} = \frac{f_1(k_1^0) - (\mu + p)k_1^0}{(\mu + p)k_2^0},$$

$$c = f_2(k_2^0)\cdot\frac{f_1(k_1^0) - (\mu + p)k_1^0}{(\mu + p)k_2^0}.$$

Proposition 1 is verified by direct computation. The results of these computations have simple geometrical interpretation.

1.2. Geometrical Description of Taut Exponential Trajectories

Let us construct the graphs of the functions $y_1 = f_1(k_1)$ and $y_1 = (\mu + p)k_1$ in a single diagram (Fig. 1a).

The ratio I_2/I_1 is equal to the ratio in which the straight line $y_1 = (\mu + p)k_1$ divides the vertical segment joining the point k_1^0 with the graph of the function f_1.

The dotted line, parallel to the straight line $y_1 = (\mu + p)k_1$, cuts off the segment OA_1, equal to $[f_1(k_1^0) - (\mu + p)k_1^0]/(\mu + p)$, on the axis of k_1.

Let us construct the graph of the function $f_2(k_2)$ and cut off a segment OA_2, equal to OA_1, on the axis of k_2 (see Fig. 1b).

The ratio L_2/L_1 is equal to the ratio of the segment OA_2 to the segment Ok_1^2.

Finally, from the similarity of triangles, we conclude that *the straight line, joining the point A_2 with the point $(k_2^0, f_2(k_2^0))$, cuts off a segment $OB_2 = c = f_2(k_2^0)[f_1(k_1^0) - (\mu + p)k_1^0]/(\mu + p)k_2^0$ on the axis of y_2.*

The problem of maximizing consumption on taut exponential trajectories is reduced to the geometrical problem of finding the points k_1^0 and k_2^0 for which the segment OB_2 is largest. For fixed segment OA_2, the segment OB_2 is largest if the dotted line in Fig. 1b touches the graph of the function $y_2 = f_2(k_2)$. This maximum increases with the increase of the segment OA_2. The segment OA_2 is largest if the dotted line in Fig. 1a touches the graph of the function $y_1 = f_1(k_1)$. Thus, for obtaining optimal k_1^m and k_2^m we must proceed in the following manner. We must draw the tangent to the graph of $y_1 = f_1(k_1)$ parallel to the straight line $y_1 = (\mu + p)k_1$. The abscissa of the point of contact is k_1^m. Next, we must cut off on the axis of k_2 a segment equal to the segment on the axis of k_1 cut off by the tangent drawn by us. From the point so obtained, we must draw a tangent to the graph of $y_2 = f_2(k_2)$. The abscissa of the point of contact is k_2^m (see Figs. 2a and 2b).

The following conditions are fulfilled for stationary values of k_1^m and k_2^m:

$$f_1'(k_1^m) = \mu + p,$$

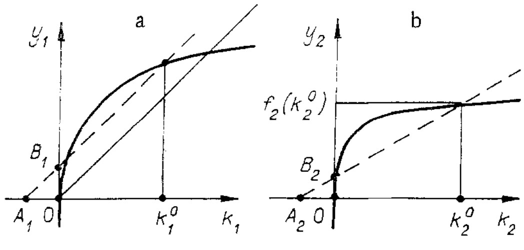

Fig. 1

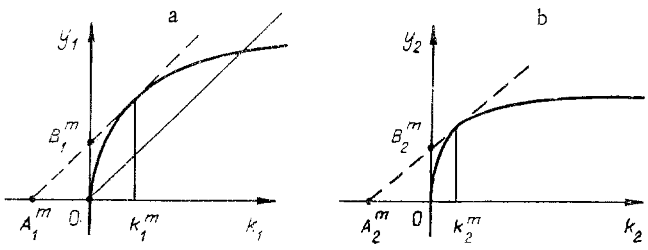

Fig. 2

$$\frac{f_1(k_1^m) - f_1'(k_1^m)k_1^m}{f_1'(k_1^m)} = \frac{f_2(k_2^m) - f_2'(k_2^m)\cdot k_2^m}{f_2'(k_2^m)}.$$

[The second of these conditions means that the tangents to the graphs of $y_1 = f_1(k)$ and $y_2 = f_2(k)$ at the points k_1^m and k_2^m, respectively, intersect on the axis of k. Let us observe that this condition is fulfilled at each moment of time for the differential-optimal trajectories of Sec. 2.]

Using these conditions and the computations of Proposition 1, we easily see that the following relations are fulfilled for the optimal stationary values:

$$\frac{I_2}{I_1} = \frac{f_1(k_1) - f_1'(k_1)k_1}{f_1'(k_1)k_1}, \tag{6}$$

$$\frac{L_2}{L_1} = \frac{f_2(k_2) - f_2'(k_2)k_2}{f_2'(k_2)k_2}. \tag{7}$$

1.3. Economic Interpretation of the Golden Trajectory

The relations (6) and (7) express the "golden rule," formulated at the beginning of this section. Let us explain it. By virtue of the homogeneity of the production functions $F_j(K_j, L_j)$, the Euler identity

$$\frac{\partial F_j}{\partial K_j}K_j + \frac{\partial F_j}{\partial L_j}L_j = F_j$$

is fulfilled. The terms on the left-hand side of equality are interpreted as the limiting profit from the **capital and the** limiting profit from the labor of the j-th sector [$\partial F_j/\partial K_j$ is the profit calculated per unit capital and $(\partial F_j/\partial K_j)\cdot K_j$ is the profit connected with the capital of the j-th sector]. We interpret $\partial F_j/\partial L_j$ and $(\partial F_j/\partial L_i)L_j$ in the same manner. The ratio $\frac{\partial F_j}{\partial L_j}L_j \Big/ \frac{\partial F_j}{\partial K_j}K_j$ of these profits is equal to the quantity

$$\frac{f_j(k_j) - f_j'(k_j)k_j}{f_j'(k_j)k_j}.$$

Remark. All the limiting profits $\partial F_j/\partial K_j$ and $\partial F_j/\partial L_j$, occurring in the "golden rule," can be expressed in terms of "observable" economic quantities such as the rates of growth of profits F_j'/F_j in each sector, the fund capacities (i.e., the ratios F_j'/K_j of funds to profits), etc. This has been done in [2] for the quantity $\partial F_1/\partial K_1$ and can be done for the remaining quantities in the same manner.

2. ASYMPTOTIC OPTIMALITY OF THE "GOLDEN TRAJECTORY"

Let c^* denote the consumption of the "golden trajectory" (see Sec. 1).

THEOREM 1. For each trajectory $x(t) = (K_1(t), L_1(t), K_2(t), L_2(t))$ of the two-sector model of economy (1)-(5) and for arbitrarily large T and arbitrarily small $\varepsilon > 0$ there exists a moment of time t_0, starting from which the average consumption during the period T does not exceed $c^* + \varepsilon$, i.e.,

$$\frac{1}{T}\int_{t_0}^{t_0+T} \frac{F_2(K_2(t), L_2(t))\,dt}{L(t)} < c^* + \varepsilon.$$

COROLLARY. The lower limit of the consumption does not exceed c^* for any trajectory.

We give a proof of Theorem 1. We start from two simple lemmas.

LEMMA 1. Let $x_s(t)$ be a trajectory of the model that depends on a parameter s and $\varphi(s)$ be finite nonnegative density with integral one: $\int \varphi(s)\, ds = 1$. Then the vector-valued function $y(t) = \int x_s(t)\varphi(s)\, ds$ is also a trajectory of the model.

Lemma 1 follows from the convexity of the differential inequalities 1)-5), controlling the dynamics in the model.

LEMMA 2. Let x(t) be a trajectory of the model. Then the vector-valued function $y(t) = \frac{1}{T}\int_t^{t+T} x(t+\tau)e^{-p\tau}d\tau$ is also a trajectory of the model.

Indeed, the displaced trajectory $x(t + \tau)$, after multiplication by $e^{-p\tau}$, is again a trajectory of the model. Lemma 2 now follows from Lemma 1.

Since the function $F_2(K_2, L_2)$ is convex and homogeneous, the average consumption on the trajectory x(t) during the period from t_0 to $t_0 + T$ is not greater than the consumption on the trajectory y(t) at the moment t_0.

Thus, to prove the theorem it is sufficient for us to show that there does not exist any trajectory on which the consumption from a certain moment exceeds $c^* + \varepsilon$.

The further arguments are based on the analysis of the two-sector model of economy given in [3].

A trajectory of the two-sector model is said to be differential-optimal if the following identities are fulfilled on it:

$$I_1 + I_2 = F_1,$$
$$\frac{\partial F_1}{\partial K_1}\cdot\frac{\partial F_2}{\partial K_2} = \frac{\partial F_1}{\partial L_1}\cdot\frac{\partial F_2}{\partial L_2}.$$

The first of these identities ensures that the whole production of the first sector goes to investments and the second identity ensures that on account of the instantaneous redistribution of the labor and the capital resources between the sectors the production cannot at once be increased in both the sectors [3].

LEMMA 3. If there exists a trajectory of the two-sector model with consumption c(t), then there also exists a differential-optimal trajectory with the same consumption.

Proof. This lemma follows immediately from the theorem on the absolute optimality of the differential-optimal trajectories in the two-sector model (see [4]).

LEMMA 4. There does not exist any differential-optimal trajectory with constant consumption greater than $c^* + \varepsilon$.

Proof. The proof of this lemma is contained in [3, Sec. 5].

Lemmas 1-4, taken together, give a proof of Theorem 1. Indeed, it follows from the existence of a trajectory with consumption greater than $c^* + \varepsilon$ that there exists a trajectory with consumption equal to $c^* + \varepsilon$, which, by virtue of Lemma 3, implies the existence of a differential-optimal trajectory with the same consumption. Lemma 4 shows that there is no such trajectory.

Proposition. For an arbitrary initial state of economy (in which the initial capital of the first sector is not equal to zero) there exists a trajectory of the model outgoing for $t \to \infty$ onto the "golden trajectory" (in the following sense: the difference of the trajectory from the "golden trajectory," divided by L(t), approaches to zero as $t \to \infty$).

Proof. As such a trajectory we can take the differential-optimal trajectory for which the labor and the capital resources are divided between the sectors in constant proportions (equal to the proportions of these quantities on the "golden trajectory"). The convergence (in the indicated sense) of similar trajectories to the exponential trajectories has been proved in [3, Sec. 3].

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BOUNDARY VALUES OF THE MAPPINGS OF THE SEMISPACE, CLOSE TO CONFORMAL ONES

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In the present paper we give sufficiently detailed proofs of statements equivalent to those announced in [1].

The successive development of the ideas generated by that approach in the investigation of the boundary behavior of the mappings $f:\mathbf{R}_+^{n+1} \to \mathbf{R}^{n+1}$ of the semispace $\mathbf{R}_+^{n+1} = \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_{n+1} > 0\}$ of the real arithmetic Euclidean space $\mathbf{R}^{n+1}$, close to conformal ones, which has been undertaken in [1], has led the author to the concept of the stability of the classes of mappings [2] containing in a natural manner the basic ingredients of the theory of stability of the conformal mappings of plane domains and of domains in multidimensional real spaces and has allowed, starting from this concept, to construct the stability theory of classes of multidimensional holomorphic mappings [3, 4]. The results of [1], which we have called to the attention of reader, will be interpreted from the point of view of the theory of the nearness of mappings to a given class of mappings [2].

1. Traces on Hyperplanes

Let n be a natural number, greater or equal to 1, and let $\mathfrak{G} = \mathfrak{G}(n, n+1)$ be the class $\mathfrak{G}_{\mathrm{VII}}$ of mappings from [2], i.e., the class of mappings of domains (open, connected sets) of the space $\mathbf{R}^n$ into the space $\mathbf{R}^{n+1}$, representing the restrictions to the considered domains of conformal affine mappings of the space $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$. We note that the conformal affine mappings are nondegenerate affine mappings preserving the angles between any two lines and coincide with the mappings $f:\mathbf{R}^n \to \mathbf{R}^{n+1}$ that are representable in the form $f(t) = aP(t) + b$, $t \in \mathbf{R}^n$, where a is a nonzero real number, $P:\mathbf{R}^n \to \mathbf{R}^{n+1}$ is a linear isometric mapping and b is a vector in the space $\mathbf{R}^{n+1}$. It is easy to verify that the class $\mathfrak{G}$ satisfies the conditions $\mathfrak{g}_1 - \mathfrak{g}_3$, $\mathfrak{g}_4'$, $\mathfrak{g}_5$, and $\mathfrak{g}_6$ of [2].

In [2] one has considered a series of nearness functionals of mappings to a given class of mappings. At the basis of the investigations, the results of which are presented in this paper, we place the functional ξ_1 from [2]. We recall the definition of this functional in connection with the case of the class $\mathfrak{G}$.

Let $f:\Delta \to \mathbf{R}^{n+1}$ be a locally bounded mapping of the domain Δ of the space $\mathbf{R}^n$ into the space $\mathbf{R}^{n+1}$ and let $B = B(x, r) = \left\{t \in \mathbf{R}^n \,\Big|\, |t - x| = \left[\sum_{j=1}^{n} (t_j - x_j)^2\right]^{1/2} < r\right\}$ be a ball of the space $\mathbf{R}^n$, contained in Δ. We set

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